3 Special Coordinate Systems

(3.1) Theorem

Let f be a coordinate system for the line ℓ in a metric geometry. If $a \in \mathbb{R}$ and ε is ± 1 and if we define $h_{a,\varepsilon} : \ell \to \mathbb{R}$ by

$$h_{a,\varepsilon}(P) = \varepsilon(f(P) - a)$$

then $h_{a,\varepsilon}$ is a coordinate system for ℓ.

1. Prove the previous theorem.

2. Let f be a coordinate system for the line ℓ in a metric geometry. Define $h_{a,\varepsilon} : \ell \to \mathbb{R}$ by $h_{a,\varepsilon}(P) = \varepsilon(f(P) - a)$ (where $a \in \mathbb{R}$, and ε is ± 1). Explain and geometrically show difference between

(i) f and $h_{0,-1}$;
(ii) f and $h_{a,1}$.

3. Let ℓ be a line in a metric geometry and let A and B be points on the line. Show that there is a coordinate system g on ℓ with $g(A) = 0$ and $g(B) > 0$.

(3.2) Definition (coordinate system with A as origin and B positive.)

Let $\ell = \ell(A, B)$. If $g : \ell \to \mathbb{R}$ is a coordinate system for ℓ with $g(A) = 0$ and $g(B) > 0$, then g is called a coordinate system with A as origin and B positive.

4. In the Euclidean Plane find a ruler f with $f(P) = 0$ and $f(Q) > 0$ for the given pair P and Q:

i. $P(2,3)$, $Q(2,-5)$; ii. $P(2,3)$, $Q(4,0)$.

5. In the Poincaré Plane find a ruler f with $f(P) = 0$ and $f(Q) > 0$ for the given pair P and Q:

i. $P(2,3)$, $Q(2,1)$; ii. $P(2,3)$, $Q(-1,6)$.

6. In the Taxicab Plane find a ruler f with $f(P) = 0$ and $f(Q) > 0$ for the given pair P and Q:

i. $P(2,3)$, $Q(2,-5)$; ii. $P(2,3)$, $Q(4,0)$.

It is reasonable to ask if there are any other operations (besides reflection and translation) that can be done to a coordinate system to get another coordinate system; that is, is every coordinate system of the form $h_{a,\varepsilon}$?

7. If ℓ is a line in a metric geometry and if $f : \ell \to \mathbb{R}$ and $g : \ell \to \mathbb{R}$ are both coordinate systems for ℓ, show that then there is an $a \in \mathbb{R}$ and an $\varepsilon = \pm 1$ with $g(P) = \varepsilon(f(P) - a)$ for all $P \in \ell$.

8. Prove that a line in a metric geometry has infinitely many points.

9. Let P and Q be points in a metric geometry. Show that there is a point M such that $M \in p(P, Q)$ and $d(P, M) = d(M, Q)$.

10. Let $\{\mathcal{S}, \mathcal{L}, d\}$ be a metric geometry and $Q \in \mathcal{S}$. If ℓ is a line through Q show that for each real number $r > 0$ there is a point $P \in \ell$ with $d(P, Q) = r$. (This says that the line really extends indefinitely.)

11. Let $g : \mathbb{R} \to \mathbb{R}$ by $g(s) = s/(|s| + 1)$. Show that g is injective.

12. Let $\{\mathcal{S}, \mathcal{L}, d\}$ be a metric geometry. For each $\ell \in \mathcal{L}$ choose a ruler f_ℓ. Define the function $\overline{d}$ by $\overline{d}(P, Q) = |g(f_\ell(P)) - g(f_\ell(Q))|$ where $\ell = \ell(P, Q)$ and g is as in Problem 11. Show that $\overline{d}$ is a distance function.

13. In Problem 12 show that $\{\mathcal{S}, \mathcal{L}, \overline{d}\}$ is not a metric geometry.

A metric geometry always has an infinite number of points (Problem 8). In particular, a finite geometry cannot be a metric geometry.